Ergodic Theory - Week 8

Course Instructor: Florian K. Richter
Teaching assistant: Konstantinos Tsinas

1 Classifying measure preserving systems

P1. **(a)** Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Show that T^k is weak mixing if and only if T is weak mixing.

(b) Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{A}, \nu, S)$ be measure-preserving systems. Show that the system $(X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu, T \times S)$ is weak-mixing if and only if both $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{A}, \nu, S)$ are weak-mixing.

P2. Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system. Prove that if the system is weak mixing then for any set $A \in \mathcal{B}$ we have

$$\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{NM} \sum_{n=1}^{N} \sum_{m=1}^{M} \mu(A \cap T^{-n}A \cap T^{-m}A \cap T^{-n-m}A) = \mu(A)^4.$$

P3. Let (X, A, μ, T) and (Y, B, ν, S) be two measure-preserving maps.

(a) Show that $T \times S$ has discrete spectrum if and only if both T and S have discrete spectrum.

(b) Give an example of a system that is neither weak mixing nor has discrete spectrum.

P4. Here, we study some ergodic theorems along subsequences under strong assumptions on the system (like weak-mixing).

(a) **(Optional)** Prove van der Corput's lemma: let $\mathcal{H}$ be a Hilbert space and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of vectors with $\|u_n\| \leq 1$. Show that if

$$\lim_{H\to+\infty} \frac{1}{H} \sum_{0\leq h\leq H} \limsup_{N\to+\infty} \left| \frac{1}{N} \sum_{1\leq n\leq N} \langle u_{n+h}, u_n \rangle \right| = 0,$$

then

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{1\leq n\leq N} u_n \right\| = 0$$

(b) Use part (a) to show that if a system $(X, \mathcal{B}, \mu, T)$ is totally ergodic (that is, T^k is ergodic for all $k \in \mathbb{N}$) then for any function $f \in L^\infty(\mu)$, we have

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^{n^2} f - \int f \, d\mu \right\|_{L^2(\mu)} = 0$$

(c) Use part (a) to show that if $(X, \mathcal{B}, \mu, T)$ is weak-mixing, then for any $f \in L^\infty(\mu)$, we have

$$\lim_{N\to+\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot T^{2n} f - \left(\int f \, d\mu \right)^2 \right\|_{L^2(\mu)} = 0$$

Conclude that if $(X, \mathcal{B}, \mu, T)$ is weak-mixing, then for any set $A \in \mathcal{B}$, we have

$$\frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A) = (\mu(A))^3 \, .$$